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Infinitely many symmetries and conservation laws for quad-graph equations via the Gardner method

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Abstract
The application of the Gardner method for the generation of conservation laws to all the ABS equations is considered. It is shown that all the necessary information for the application of the Gardner method, namely Bäcklund transformations and initial conservation laws, follows from the multidimensional consistency of ABS equations. We also apply the Gardner method to an asymmetric equation which is not included in the ABS classification. An analog of the Gardner method for the generation of symmetries is developed and applied to the discrete Korteweg–de Vries equation. It can also be applied to all the other ABS equations.

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1. Introduction

The first integrable example of a partial difference equation (PΔE) goes back to [4], in which Hirota wrote down an equation on a quad-graph related by a simple change of variables to the discrete Korteweg–de Vries (KdV) equation. However, the subject has only been intensively developed in the last few years. A particular milestone was the so-called ABS classification of integrable scalar quad-graph equations based on the principle of 'consistency on the cube' [1]. All the equations in this classification have the discrete analog of a matrix Lax pair and also a 'natural' auto-Bäcklund transformation [2]. Other integrability properties (discrete analogs of scalar Lax pairs, bi-Hamiltonian structures and infinite numbers of conservation laws) have been studied less.

The topic of conservation laws for the PΔE has become very important recently. This started in the work of Orphanidis [12] in which he presents conservation laws for the fully discrete sine-Gordon equation. The next step in the research of conservation laws was the introduction of a systematic computation method. The first systematic method developed was the direct method [6, 15, 16]. This method allows use of computer algebra, which makes it more constructive. But it has some disadvantages: it requires massive computations and cannot produce infinite numbers of conservation laws. The next method for computing conservation laws was proposed in [17]. It produces infinite numbers of conservation laws by acting with mastersymmetry on basic conservation laws.

A new approach to the computation of conservation laws for PΔE has appeared very recently [14]. It is called the Gardner method and is an analog of the Gardner method for partial differential equations. The method uses a Bäcklund transformation (BT) to generate conservation laws. It seems that the Gardner method is the most efficient among the methods described above.

Symmetries for PΔE were also researched recently. They first appeared as similarity constraints for integrable lattices [10]. Tongas *et al* pointed out that the similarity constraints for quad-graph equations obtained previously are equivalent to the characteristics of symmetries [20]. Symmetries of several quad-graph equations have been found in [7, 8, 13, 19]. Hydon developed a direct method for finding symmetries based on solving functional equations by creating an associated system of differential equations that can be solved [5, 6]. In [18] this method was applied to PΔE. There the authors computed five-point symmetries for all the ABS equations (equations from [1]). Symmetries for PΔE were also discussed in other articles [9, 21].

Our intention is to show that the Gardner method for the generation of conservation laws for PΔE can be applied systematically. We apply it to all the ABS equations and to an asymmetric quad-graph equation. We also show that the Gardner method can be applied for symmetries. Namely, we use it to generate an infinite number of symmetries from the known one.

2. The Gardner method

Before presenting the Gardner method for generating conservation laws for PΔE, we review the method for the continuum KdV's equation. The Gardner method for the KdV equation starts with the BT. The BT states that if u solves the KdV equation, then so does $u + v_x$ where v is a solution of the system

$$\begin{aligned} v_x &= \theta - 2u - v^2 \\ v_t &= -\tfrac{1}{2}u_{xx} + (\theta + u)(\theta - 2u - v^2) + u_x v. \end{aligned}$$

Here, θ is a parameter. It is straightforward to check that v is actually the G component of a conservation law. Specifically, we have

$$\partial_t v + \partial_x \left(\tfrac{1}{2}u_x - (u+\theta)v\right) = 0.$$

But v depends on the parameter θ. Expanding in a suitable series in θ yields an infinite number of conservation laws. For the KdV equation the appropriate series is

$$v = \theta^{1/2} - \frac{u}{\theta^{1/2}} + \frac{u_x}{2\theta} - \frac{u_{xx} + 2u^2}{4\theta^{3/2}} + \frac{u_{xxx} + 8uu_x}{8\theta^2} - \frac{u_{xxxx} + 8u^3 + 10u_x^2 + 12uu_{xx}}{16\theta^{5/2}} + O(\theta^{-3}).$$

In [14] the implementation of the Gardner method to the dKdV equation was given.

Here we apply the Gardner method to the following asymmetric quad-graph equation [11]:

$$\alpha u_{0,0}u_{1,0} - \beta(u_{0,0}u_{0,1} + u_{1,0}u_{1,1}) = 0. \tag{1}$$

Here $k, l \in \mathbb{Z}^2$ are independent variables and $u_{0,0} = u(k,l)$ is a dependent variable that is defined on the domain $\mathbb{Z}^2$. We denote the values of this variable on other points by $u_{i,j} = u(k+i, l+j) = S_k^i S_l^j u_{0,0}$, where S_k, S_l are the unit forward shift operators in k and l

respectively. Equation (1) is not included in the ABS list, since it is asymmetric. A BT for (1) is

$$\alpha(u_{0,0}u_{1,0} + \tilde{u}_{0,0}\tilde{u}_{1,0}) - \theta(u_{0,0}\tilde{u}_{0,0} + u_{1,0}\tilde{u}_{1,0}) = 0, \tag{2}$$

$$\theta u_{0,0}\tilde{u}_{0,0} - \beta(u_{0,0}u_{0,1} + \tilde{u}_{0,0}\tilde{u}_{0,1}) = 0. \tag{3}$$

Here θ is a parameter. This BT follows from the multidimensional consistency of equations (1)–(3) embedded in three dimensions. The consistency of equations (1)–(3) was already considered in [2]. It corresponds to the $H3(0)$ to $H3(\delta)$ transformation in the special case $\delta = 0$. Let us introduce the third variable m so that

$$u_{k,l,m} = u_{0,0,0} = u_{0,0}, \qquad u_{0,0,1} = \tilde{u}_{0,0}, \qquad u_{1,0,1} = \tilde{u}_{1,0}, \qquad \text{etc.} \tag{4}$$

Then, equations (1)–(3) can be written as

$$\begin{aligned} P_1 &: \quad \alpha u_{0,0,0}u_{1,0,0} - \beta(u_{0,0,0}u_{0,1,0} + u_{1,0,0}u_{1,1,0}) = 0, \\ P_2 &: \quad \alpha(u_{0,0,0}u_{1,0,0} + u_{0,0,1}u_{1,0,1}) - \theta(u_{0,0,0}u_{0,0,1} + u_{1,0,0}u_{1,0,1}) = 0, \\ P_3 &: \quad \theta u_{0,0,0}u_{0,0,1} - \beta(u_{0,0,0}u_{0,1,0} + u_{0,0,1}u_{0,1,1}) = 0. \end{aligned} \tag{5}$$

Equations P_2 and P_3 form a BT for P_1. In what follows we use the notation (4), writing $u_{0,0,1}$ and $u_{0,0,-1}$ for the BT and inverse BT of $u_{0,0,0}$, where possible; however, we revert to two-dimensional notation.

In general we cannot solve the equations of the BT to write $u_{0,0,1}$ in terms of $u_{0,0,0}$. However, there is a special case $\theta = \alpha$ for which $u_{0,0,1} = u_{1,0,0}$ or $u_{-1,0,0}$. Consider $\theta = \alpha + \epsilon$ where ϵ is small, and look for a solution of the BT in the form

$$u_{0,0,1} = u_{1,0} + \sum_{i=1}^{\infty} v_{0,0}^{(i)}\epsilon^i. \tag{6}$$

We just look at the first equation of the BT. This reads

$$\sum_{i=1}^{\infty} v_{0,0}^{(i)}\epsilon^i \left(\alpha u_{2,0} + \alpha\sum_{i=1}^{\infty} v_{1,0}^{(i)}\epsilon^i - \alpha u_{0,0} - \epsilon u_{0,0}\right) = \epsilon u_{1,0}\left(u_{0,0} + u_{2,0} + \sum_{i=1}^{\infty} v_{1,0}^{(i)}\epsilon^i\right).$$

The leading-order approximation gives

$$v_{0,0}^{(1)} = \frac{u_{1,0}(u_{0,0} + u_{2,0})}{\alpha(u_{2,0} - u_{0,0})}.$$

Higher order terms give

$$v_{0,0}^{(i)} = \frac{u_{0,0}v_{0,0}^{(i-1)} + u_{1,0}v_{1,0}^{(i-1)} - \alpha\sum_{j=1}^{i-1} v_{0,0}^{(i)}v_{0,0}^{(j-i)}}{\alpha(u_{2,0} - u_{0,0})}, \qquad i = 2, 3 \ldots. \tag{7}$$

As in the case of the dKdV equation all these formulas are on the horizontal line, i.e. all the $v_{0,0}^{(i)}$ only depend on values of $u_{i,j}$ with $j = 0$. An infinite sequence of conservation laws can be obtained starting from the ϵ expansion of a single conservation law. It is straightforward to check that if we define

$$F = \ln\left(\frac{u_{0,0,1}}{u_{0,0,0}}\right), \qquad G = -\ln\left(\frac{\alpha u_{0,0,1} - \theta u_{1,0,0}}{u_{0,0,0}}\right), \tag{8}$$

then

$$(S_k - I)F + (S_l - I)G = 0,$$

on solutions of (5). By plugging (6) and $\theta = \alpha + \epsilon$ into (8) and expanding $F = \sum_{i=0}^{\infty} F_i \epsilon^i$ and $G = -\ln(2\epsilon) + \sum_{i=0}^{\infty} G_i \epsilon^i$ we obtain

$$F_0 = \ln\left(\frac{u_{1,0}}{u_{0,0}}\right), \qquad F_1 = \frac{u_{2,0} + u_{0,0}}{u_{2,0} - u_{0,0}},$$

$$G_0 = -\ln\left(\frac{u_{1,0}}{u_{2,0} - u_{0,0}}\right), \qquad G_1 = \frac{u_{1,0}(3u_{2,0} + u_{0,0}) + u_{3,0}(u_{2,0} - u_{0,0})}{2u_{0,0}(u_{3,0} - u_{0,0})(u_{2,0} - u_{0,0})},$$

$$F_2 = \frac{4u_{2,0}u_{0,0}(u_{3,0} + u_{1,0}) + \left(u_{2,0}^2 - u_{0,0}^2\right)(u_{3,0} - u_{1,0})}{2(u_{2,0} - u_{0,0})^2(u_{1,0} - u_{3,0})},$$

$$G_2 = \frac{(u_{3,0}u_{2,0} + 3u_{2,0}u_{1,0} + u_{1,0}u_{0,0} - u_{3,0}u_{0,0})^2}{8(u_{2,0} - u_{0,0})^2(u_{3,0} - u_{1,0})^2}, \quad \text{etc.}$$

Thus, we see how the expansion of the BT around the point $\theta = \alpha$ yields an infinite sequence of conservation laws on the horizontal line. Expansion around $\theta = \beta$ does not seem to yield an infinite sequence of conservation laws on the vertical line, since (1) is asymmetric. In the case of the dKdV equation expansion around $\theta = \beta$ also gives an infinite number of conservation laws.

3. The Gardner method for conservation laws of integrable equations on the quad-graph

In this section we present all the necessary information for the application of the Gardner method to the quad-graphs that are listed in [1]. The general form of ABS equations is

$$P_1\colon \quad P(u_{0,0,0}, u_{1,0,0}, u_{0,1,0}, u_{1,1,0}, \alpha, \beta) = 0. \tag{9}$$

For the application of the Gardner method we need a special BT and the initial conservation law (ICL). For all the ABS equations the necessary BTs are known and these are so-called natural auto-BTs [2]. The general form of the natural auto-BTs for ABS equations can be obtained from the form of equation (9). For (9) the form of the natural auto-BT is

$$P_2\colon \quad P(u_{0,0,0}, u_{1,0,0}, u_{0,0,1}, u_{1,0,1}, \alpha, \theta) = 0, \tag{10}$$

$$P_3\colon \quad P(u_{0,0,0}, u_{0,0,1}, u_{0,1,0}, u_{0,1,1}, \theta, \beta) = 0. \tag{11}$$

This is the BT which we need for the application of the Gardner method. Note that equations (9)–(11) are one equation embedded in three dimensions.

The ICL also follows from multidimensional consistency. Expressions for conservation laws for P_1, P_2, P_3 are, respectively,

$$\begin{aligned} &(S_k - I)F_1 + (S_l - I)G_1 = 0, \\ &(S_k - I)F_2 + (S_m - I)H_2 = 0, \\ &(S_l - I)G_3 + (S_m - I)H_3 = 0. \end{aligned} \tag{12}$$

Five-point conservation laws for ABS equations were found in [17], where the authors show that each ABS equation has three five-point conservation laws. Two five-point conservation laws for P_1 can always be presented as

$$F = f(u_{0,-1,0}, u_{0,0,0}, u_{0,1,0}, \beta), \qquad G = g(u_{0,-1,0}, u_{0,0,0}, u_{1,0,0}, \alpha, \beta), \tag{13}$$

$$F' = g(u_{-1,0,0}, u_{0,0,0}, u_{0,1,0}, \beta, \alpha), \qquad G' = f(u_{-1,0,0}, u_{0,0,0}, u_{1,0,0}, \alpha). \tag{14}$$

As we said before, equations P_1, P_2, P_3 are one equation which is embedded in three dimensions. Therefore, conservation laws for P_2, P_3 can be obtained from conservation laws for P_1 by rewriting them on the appropriate plane. The conservation law for P_2 which corresponds to F, G is

$$F_2 = f(u_{0,0,-1}, u_{0,0,0}, u_{0,0,1}, \theta), \qquad H_2 = g(u_{0,0,-1}, u_{0,0,0}, u_{1,0,0}, \alpha, \theta). \tag{15}$$

The conservation law for P_3 which corresponds to F, G is

$$G_3 = f(u_{0,0,-1}, u_{0,0,0}, u_{0,0,1}, \theta), \qquad H_3 = g(u_{0,0,-1}, u_{0,0,0}, u_{0,1,0}, \beta, \theta). \tag{16}$$

F_2 is equal to G_3, so with the help of a linear combination of expressions from (12) we obtain that

$$\begin{aligned}(S_l - I)(S_k - I)F_2 + (S_l - I)(S_m - I)H_2 - (S_k - I)(S_l - I)G_3 \\ - (S_k - I)(S_m - I)H_3 = (S_m - I)((S_l - I)H_2 - (S_k - I)H_3) = 0\end{aligned}$$

is true on solutions of P_2 and P_3. This expression can be integrated with respect to m to give

$$(S_l - I)H_2 - (S_k - I)H_3 = C(k, l) \tag{17}$$

where $C(k, l)$ is a function which has to be determined. Equations P_2 and P_3 are symmetric with respect to the reflection of the coordinate m, that is

$$\begin{aligned}P(u_{0,0,0}, u_{1,0,0}, u_{0,0,1}, u_{1,0,1}, \alpha, \theta) &= \pm P(u_{0,0,1}, u_{1,0,1}, u_{0,0,0}, u_{1,0,0}, \alpha, \theta) \\ &= \pm S_m P(u_{0,0,0}, u_{1,0,0}, u_{0,0,-1}, u_{1,0,-1}, \alpha, \theta).\end{aligned}$$

Therefore, (17) is also symmetric with respect to the reflection of coordinate m, so

$$(S_l - I)g(u_{0,0,1}, u_{0,0,0}, u_{1,0,0}, \alpha, \theta) - (S_k - I)g(u_{0,0,1}, u_{0,0,0}, u_{0,1,0}, \beta, \theta) = C(k, l). \tag{18}$$

By substitution of P_2 and P_3 into (18) we obtain that $C(k, l) = 0$ for all the ABS equations. Therefore,

$$(S_l - I)H_2 - (S_k - I)H_3 = 0,$$

for all the ABS equations. Thus, the components of ICLs for all the ABS equations are

$$F_{\mathrm{ICL}} = -g(u_{0,0,1}, u_{0,0,0}, u_{0,1,0}, \beta, \theta), \quad G_{\mathrm{ICL}} = g(u_{0,0,1}, u_{0,0,0}, u_{1,0,0}, \alpha, \theta).$$

We present a table with ICLs for all the ABS equations below. The equations from the ABS classification are as follows: for convenience, we have used the form of **Q4** that was discovered by Hietarinta [3].

$$\begin{aligned}
&\mathbf{Q1}: \quad \alpha(u_{0,0} - u_{0,1})(u_{1,0} - u_{1,1}) - \beta(u_{0,0} - u_{1,0})(u_{0,1} - u_{1,1}) + \delta^2\alpha\beta(\alpha - \beta) = 0, \\
&\mathbf{Q2}: \quad \alpha(u_{0,0} - u_{0,1})(u_{1,0} - u_{1,1}) - \beta(u_{0,0} - u_{1,0})(u_{0,1} - u_{1,1}) \\
&\qquad + \alpha\beta(\alpha - \beta)(u_{0,0} + u_{1,0} + u_{0,1} + u_{1,1}) - \alpha\beta(\alpha - \beta)(\alpha^2 - \alpha\beta + \beta^2) = 0, \\
&\mathbf{Q3}: \quad (\beta^2 - \alpha^2)(u_{0,0}u_{1,1} + u_{1,0}u_{0,1}) + \beta(\alpha^2 - 1)(u_{0,0}u_{1,0} + u_{0,1}u_{1,1}) \\
&\qquad - \alpha(\beta^2 - 1)(u_{0,0}u_{0,1} + u_{1,0}u_{1,1}) - \delta^2(\alpha^2 - \beta^2)(\alpha^2 - 1)(\beta^2 - 1)/(4\alpha\beta) = 0, \\
&\mathbf{Q4}: \quad \mathrm{sn}(\alpha)(u_{0,0}u_{1,0} + u_{0,1}u_{1,1}) - \mathrm{sn}(\beta)(u_{0,0}u_{0,1} + u_{1,0}u_{1,1}) - \mathrm{sn}(\alpha - \beta)(u_{0,0}u_{1,1} + u_{1,0}u_{0,1}) \\
&\qquad + \mathrm{sn}(\alpha - \beta)\mathrm{sn}(\alpha)\mathrm{sn}(\beta)(1 + K^2 u_{0,0}u_{1,0}u_{0,1}u_{1,1}) = 0, \\
&\mathbf{H1}: \quad (u_{0,0} - u_{1,1})(u_{1,0} - u_{0,1}) + \beta - \alpha = 0, \\
&\mathbf{H2}: \quad (u_{0,0} - u_{1,1})(u_{1,0} - u_{0,1}) + (\beta - \alpha)(u_{0,0} + u_{1,0} + u_{0,1} + u_{1,1}) + \beta^2 - \alpha^2 = 0, \\
&\mathbf{H3}: \quad \alpha(u_{0,0}u_{1,0} + u_{0,1}u_{1,1}) - \beta(u_{0,0}u_{0,1} + u_{1,0}u_{1,1}) + \delta^2(\alpha^2 - \beta^2) = 0, \\
&\mathbf{A1}: \quad \alpha(u_{0,0} + u_{0,1})(u_{1,0} + u_{1,1}) - \beta(u_{0,0} + u_{1,0})(u_{0,1} + u_{1,1}) - \delta^2\alpha\beta(\alpha - \beta) = 0, \\
&\mathbf{A2}: \quad (\beta^2 - \alpha^2)(u_{0,0}u_{1,0}u_{0,1}u_{1,1} + 1) + \beta(\alpha^2 - 1)(u_{0,0}u_{0,1} + u_{1,0}u_{1,1}) \\
&\qquad - \alpha(\beta^2 - 1)(u_{0,0}u_{1,0} + u_{0,1}u_{1,1}) = 0.
\end{aligned} \tag{19}$$

Table 1. Initial conservation laws for the equations from the ABS classification.

Equation	Components
Q1	$F = \ln(-\tilde{u}_{0,0} + u_{0,0} - \delta\theta) - \ln(\delta(\beta - \theta) + u_{0,1} - \tilde{u}_{0,0})$, $G = -\ln(u_{0,0} - \tilde{u}_{0,0} - \delta\theta) + \ln(\delta(\alpha - \theta) + u_{1,0} - \tilde{u}_{0,0})$
Q2	$F = \ln((\theta\beta^2 + \beta(\tilde{u}_{0,0} - u_{0,0} - \theta^2) + \theta(u_{0,0} - u_{0,1}))^2) - \ln(\beta^4 - 2\beta^2(u_{0,1} + u_{0,0}) + (u_{0,0} - u_{0,1})^2)$, $G = -\ln((\theta\alpha^2 + \alpha(\tilde{u}_{0,0} - u_{0,0} - \theta^2) + \theta(u_{0,0} - u_{1,0}))^2) + \ln(\alpha^4 - 2\alpha^2(u_{1,0} + u_{0,0}) + (u_{0,0} - u_{1,0})^2)$
Q3	$F = \ln((\theta^2(u_{0,0} - \beta u_{0,1}))^2 - \theta(1 - \alpha^2)\tilde{u}_{0,0} - \alpha^2 u_{0,0} + \alpha^2 u_{0,1}) - \ln(4\beta(\beta u_{0,0} - u_{0,1})(u_{0,0} - \beta u_{0,1}) + \delta(1 - \beta^2)^2)$, $G = -\ln((\theta^2(u_{0,0} - \alpha u_{1,0}) - \theta\tilde{u}_{0,0}(1 - \alpha^2) - \alpha^2 u_{0,0} + \alpha u_{1,0})^2) + \ln(4\alpha(\alpha u_{0,0} - u_{1,0})(u_{0,0} - \alpha u_{1,0}) + \delta(1 - \alpha^2)^2)$
Q4	$F = \ln(\mathrm{sn}(\theta)^2(1 + K^2 u_{0,0}\tilde{u}_{0,0}) + 2\mathrm{cn}(\theta)\mathrm{dn}(\theta)u_{0,0}\tilde{u}_{0,0} - u_{0,0}^2 - \tilde{u}_{0,0}^2) - \ln(\mathrm{sn}(\theta - \beta)^2(u_{0,1}\tilde{u}_{0,0} - \mathrm{sn}(\theta)\mathrm{sn}(\beta))(\mathrm{sn}(\theta)\mathrm{sn}(\beta)K^2\tilde{u}_{0,0}u_{0,1} - 1) - (\mathrm{sn}(\theta)u_{0,1} - \mathrm{sn}(\beta)\tilde{u}_{0,0})(\mathrm{sn}(\beta)u_{0,1} - \mathrm{sn}(\theta)\tilde{u}_{0,0}))$ $G = -\ln(\mathrm{sn}(\theta)^2(1 + K^2 u_{0,0}\tilde{u}_{0,0}) + 2\mathrm{cn}(\theta)\mathrm{dn}(\theta)u_{0,0}\tilde{u}_{0,0} - u_{0,0}^2 - \tilde{u}_{0,0}^2) + \ln(\mathrm{sn}(\alpha - \theta)^2(u_{1,0}\tilde{u}_{0,0} - \mathrm{sn}(\theta)\mathrm{sn}(\alpha))(\mathrm{sn}(\theta)\mathrm{sn}(\alpha)K^2\tilde{u}_{0,0}u_{1,0} - 1) - (\mathrm{sn}(\theta)u_{1,0} - \mathrm{sn}(\alpha)\tilde{u}_{0,0})(\mathrm{sn}(\alpha)u_{1,0} - \mathrm{sn}(\theta)\tilde{u}_{0,0}))$,
H1	$F = -\ln(\tilde{u}_{0,0} - u_{0,1})$, $G = \ln(\tilde{u}_{0,0} - u_{1,0})$
H2	$F = \ln\left((\beta - \theta + u_{0,1} - \tilde{u}_{0,0})^2\right) - \ln(\beta + u_{0,0} + u_{0,1})$, $G = -\ln\left((\alpha - \theta + u_{1,0} - \tilde{u}_{0,0})^2\right) + \ln(\alpha + u_{0,0} + u_{1,0})$
H3	$F = \ln\left((\beta\tilde{u}_{0,0} - \theta u_{0,1})^2\right) - \ln(\delta\beta + u_{0,0}u_{0,1})$, $G = -\ln\left((\alpha\tilde{u}_{0,0} - \theta u_{1,0})^2\right) + \ln(\delta\alpha + u_{0,0}u_{1,0})$
A1	$F = \ln((u_{0,0} + \tilde{u}_{0,0})^2 - \delta\theta^2) - \ln((u_{0,1} - \tilde{u}_{0,0})^2 - \delta(\theta - \beta)^2)$, $G = -\ln((u_{0,0} + \tilde{u}_{0,0})^2 - \delta\theta^2) + \ln((u_{1,0} - \tilde{u}_{0,0})^2 - \delta(\alpha - \theta)^2)$
A2	$F = \ln(\tilde{u}_{0,0}u_{0,0} - \theta)(\theta\tilde{u}_{0,0}u_{0,0} - 1) - \ln((\theta u_{0,1} - \beta\tilde{u}_{0,0})(\beta u_{0,1} - \theta\tilde{u}_{0,0}))$, $G = -\ln(\tilde{u}_{0,0}u_{0,0} - \theta)(\theta\tilde{u}_{0,0}u_{0,0} - 1) + \ln((\theta u_{1,0} - \alpha\tilde{u}_{0,0})(\alpha u_{1,0} - \theta\tilde{u}_{0,0}))$

Here $\mathrm{sn}(\alpha) = \mathrm{sn}(\alpha; K)$ is a Jacobi elliptic function with modulus K. Without loss of generality, the parameter δ is restricted to the values 0 and 1. Obtained ICLs are summarized in table 1, in which we list the components F and G for each of the ABS equations.

Another problem which can appear during the application of the Gardner method is that $v_{0,0}^{(i)}$ has to be found for $i = 1, 2, \ldots$ and for all the ABS equations. We prove that it is always possible to explicitly find $v_{0,0}^{(i)}$ for $i = 1, 2, \ldots$ for all the ABS equations in the application of the Gardner method in the k direction. For the l direction the proof is similar. Consider $\theta = \alpha + \epsilon$ where ϵ is small, and look for a solution of the BT in the form

$$u_{0,0,1} = u_{1,0} + \sum_{i=1}^{\infty} v_{0,0}^{(i)}\epsilon^i. \tag{20}$$

We look only at the first equation of BT. This reads

$$P\left(u_{0,0}, u_{1,0}, u_{1,0} + \sum_{i=1}^{\infty} v_{0,0}^{(i)}\epsilon^i, u_{2,0} + \sum_{i=1}^{\infty} v_{1,0}^{(i)}\epsilon^i, \alpha, \alpha + \epsilon\right) = 0. \tag{21}$$

Let C_n be the coefficient next to the ϵ^n in the expansion of (21) in the Taylor series around $\epsilon = 0$. At first sight C_n depends upon $v_{0,0}^{(i)}, v_{1,0}^{(i)}, i = 1, 2, \ldots, n$, and to find $v_{0,0}^{(i)}$ one has to solve non-trivial difference equations. We show that C_n depends just upon $v_{0,0}^{(i)}, v_{1,0}^{(j)}, i = 1, 2, \ldots, n, j = 1, 2, \ldots, n-1$, so $C_n = 0$ can be solved explicitly with respect to $v_{0,0}^{(n)}$. First of all let us show

$$C_0 = P(u_{0,0}, u_{1,0}, u_{1,0}, u_{2,0}, \alpha, \alpha) = 0.$$

Since P is an ABS equation it has the symmetry property [1]

$$P(x, u, v, y, \alpha, \beta) = -P(x, v, u, y, \beta, \alpha).$$

According to this symmetry property we obtain

$$P(u_{0,0}, u_{1,0}, u_{1,0}, u_{2,0}, \alpha, \alpha) = -P(u_{0,0}, u_{1,0}, u_{1,0}, u_{2,0}, \alpha, \alpha) = 0.$$

The derivative of (21) with respect to ϵ is

$$\frac{\mathrm{d}}{\mathrm{d}\epsilon}P = \sum_{i=1}^{\infty} i v_{0,0}^{(i)} \epsilon^{i-1} P_3 + \sum_{i=1}^{\infty} i v_{1,0}^{(i)} \epsilon^{i-1} P_4 + P_6, \tag{22}$$

where P_i is the derivative of P with respect to its ith argument. For example

$$P_3 = \frac{\partial}{\partial a} P\left(u_{0,0}, u_{1,0}, a, u_{2,0} + \sum_{i=1}^{\infty} v_{1,0}^{(i)} \epsilon^i, \alpha, \alpha + \epsilon\right)\Bigg|_{a=u_{1,0}+\sum_{i=1}^{\infty} v_{0,0}^{(i)}\epsilon^i}.$$

For $\epsilon = 0$ we obtain C_1

$$\begin{aligned} C_1 = \frac{\mathrm{d}}{\mathrm{d}\epsilon}P\bigg|_{\epsilon=0} &= v_{0,0}^{(1)} P_3(u_{0,0}, u_{1,0}, u_{1,0}, u_{2,0}, \alpha, \alpha) + v_{1,0}^{(1)} P_4(u_{0,0}, u_{1,0}, u_{1,0}, u_{2,0}, \alpha, \alpha) \\ &\quad + P_6(u_{0,0}, u_{1,0}, u_{1,0}, u_{2,0}, \alpha, \alpha). \end{aligned} \tag{23}$$

We checked that for all the ABS equations

$$P_3(u_{0,0}, u_{1,0}, u_{1,0}, u_{2,0}, \alpha, \alpha) \neq 0, \qquad P_4(u_{0,0}, u_{1,0}, u_{1,0}, u_{2,0}, \alpha, \alpha) = 0.$$

So, C_1 depends upon $v_{0,0}^{(1)}$ and does not depend upon $v_{1,0}^{(1)}$. From (22) it is seen that generally the form of C_n is similar to the form of C_1, namely

$$C_n = n! v_{0,0}^{(n)} P_3(u_{0,0}, u_{1,0}, u_{1,0}, u_{2,0}, \alpha, \alpha) + n! v_{1,0}^{(n)} P_4(u_{0,0}, u_{1,0}, u_{1,0}, u_{2,0}, \alpha, \alpha) + R.$$

Here R does not depend upon $v_{0,0}^{(n)}, v_{1,0}^{(n)}$. As we said before $P_4(u_{0,0}, u_{1,0}, u_{1,0}, u_{2,0}, \alpha, \alpha) = 0$; therefore, $C_n = 0$ can be solved with respect to $v_{0,0}^{(n)}$. By solving C_i with respect to $v_{0,0}^{(i)}$ iteratively for $i = 1, 2, \ldots$ we can find as many $v_{0,0}^{(i)}$ as necessary. Once $v_{0,0}^{(i)}, i = 1, \ldots, n$, are known, we plug (20) into the ICL and expand it around $\epsilon = 0$ up to order n. In this way n conservation laws can be found.

4. The Gardner method for the symmetries of integrable equations on the quad-graph

In [18] five-point symmetries were found for ABS equations. The authors show there that each ABS equation P_1 has four five-point symmetries. Two of these symmetries can always be presented as

$$X_h = \eta(u_{-1,0,0}, u_{0,0,0}, u_{1,0,0}, \alpha)\partial_{u_{0,0,0}}, \tag{24}$$

$$X_v = \eta(u_{0,-1,0}, u_{0,0,0}, u_{0,1,0}, \beta)\partial_{u_{0,0,0}}. \tag{25}$$

Therefore, equations P_2 and P_3 both have a symmetry

$$X = \eta(u_{0,0,-1}, u_{0,0,0}, u_{0,0,1}, \theta)\partial_{u_{0,0,0}}. \tag{26}$$

Since P_1, P_2 and P_3 are consistent in three dimensions, we obtain that X is also a symmetry for P_1. Let us call X the *initial symmetry* for P_1. Let us apply the Gardner method to X or, in other words, expand X in a series. X involves the BT and inverse BT $u_{0,0,1}$ and $u_{0,0,-1}$; therefore, we are looking for the solution of those for $\theta = \alpha \pm \epsilon$ where ϵ is small in the forms

$$u_{0,0,1} = u_{1,0,0} + \sum_{i=1}^{\infty} v_{0,0}^{(i)}\epsilon^i, \tag{27}$$

$$u_{0,0,-1} = u_{-1,0,0} + \sum_{i=1}^{\infty} w_{0,0}^{(i)}\epsilon^i. \tag{28}$$

By plugging this in the characteristic η of the symmetry X we obtain

$$\eta = \eta\left(u_{-1,0,0} + \sum_{i=1}^{\infty} w_{0,0}^{(i)}\epsilon^i, u_{0,0,0}, u_{1,0,0} + \sum_{i=1}^{\infty} v_{0,0}^{(i)}\epsilon^i, \alpha + \epsilon\right). \tag{29}$$

After expansion of η in Taylor series around $\epsilon = 0$ we obtain

$$\eta = \sum_{i=0}^{\infty} \eta_i \epsilon^i. \tag{30}$$

Each η_i, $i = 1, 2, \ldots$, is the characteristic of a symmetry for P_1. For example let us consider the discrete KdV equation

$$(u_{0,0,0} - u_{1,1,0})(u_{1,0,0} - u_{0,1,0}) + \beta - \alpha = 0. \tag{31}$$

Two of the symmetries for this equation are

$$X_h = \frac{1}{u_{1,0,0} - u_{-1,0,0}}\partial_{u_{0,0,0}}, \tag{32}$$

$$X_v = \frac{1}{u_{0,1,0} - u_{0,-1,0}}\partial_{u_{0,0,0}}. \tag{33}$$

Therefore,

$$X = \frac{1}{u_{0,0,1} - u_{0,0,-1}}\partial_{u_{0,0,0}} \tag{34}$$

is also a symmetry. Expression (29) for X is

$$\eta = \frac{1}{u_{1,0,0} - u_{-1,0,0} + \sum_{i=1}^{\infty}\left(v_{0,0}^{(i)} - w_{0,0}^{(i)}\right)\epsilon^i}.$$

We just look at the first equation of the BT which is P_2. This reads

$$\epsilon = \left(\sum_{i=1}^{\infty} v_{0,0}^{(i)}\epsilon^i\right)\left(u_{0,0,0} - u_{2,0,0} - \sum_{i=1}^{\infty} v_{1,0}^{(i)}\epsilon^i\right), \tag{35}$$

$$\epsilon = \left(\sum_{i=1}^{\infty} w_{1,0}^{(i)}\epsilon^i\right)\left(u_{-1,0,0} - u_{1,0,0} + \sum_{i=1}^{\infty} w_{0,0}^{(i)}\epsilon^i\right). \tag{36}$$

The leading-order approximation gives

$$v_{0,0}^{(1)} = \frac{1}{u_{0,0,0} - u_{2,0,0}}, \tag{37}$$

$$w_{1,0}^{(1)} = \frac{1}{u_{-1,0,0} - u_{1,0,0}}. \tag{38}$$

Higher order terms give

$$v_{0,0}^{(i)} = \frac{1}{u_{0,0,0} - u_{2,0,0}} \sum_{j=1}^{i-1} v_{0,0}^{(j)} v_{1,0}^{(i-j)}, \qquad i = 2, 3, \ldots, \tag{39}$$

$$w_{1,0}^{(i)} = -\frac{1}{u_{-1,0,0} - u_{1,0,0}} \sum_{j=1}^{i-1} w_{0,0}^{(j)} w_{1,0}^{(i-j)}, \qquad i = 2, 3, \ldots. \tag{40}$$

The first three coefficients in the expansion of η around $\epsilon = 0$ are

$$\begin{aligned}
\eta_1 &= \frac{1}{u_{1,0} - u_{-1,0}}, \\
\eta_2 &= \frac{u_{-2,0} - u_{2,0}}{(u_{-1,0} - u_{1,0})^2 (u_{0,0} - u_{2,0})(u_{0,0} - u_{-2,0})}, \\
\eta_3 &= \frac{u_{-1,0} - u_{3,0}}{(u_{0,0} - u_{2,0})^2 (u_{1,0} - u_{3,0})(u_{-1,0} - u_{1,0})^3} + \frac{2}{(u_{0,0} - u_{2,0})(u_{-2,0} - u_{2,0})(u_{-1,0} - u_{1,0})^3} \\
&\quad + \frac{u_{-3,0} - u_{1,0}}{(u_{0,0} - u_{-2,0})^2 (u_{-3,0} - u_{-1,0})(u_{-1,0} - u_{1,0})^3} \\
&\quad - \frac{2}{(u_{0,0} - u_{-2,0})(u_{-2,0} - u_{2,0})(u_{-1,0} - u_{1,0})^3}.
\end{aligned}$$

These are characteristics of symmetries for (31). η_1 and η_2 were already presented in [20].

5. Concluding remarks

In this paper we showed how to apply the Gardner method for the generation of conservation laws to all the ABS equations. It is shown that all the necessary information for the application of the Gardner method follows from the multidimensional consistency of ABS equations. Namely the natural auto-BT for the ABS equation is the equation itself embedded in three dimensions. This construction is possible because of the multidimensional consistency of the equation. The ICL for ABS equations follows from the linear combinations of conservation laws for the corresponding embedded equations.

Another important result of this paper is the introduction of the Gardner method for symmetry generation for ABS equations. In the Gardner method for the symmetry generation one has to substitute expressions

$$u_{0,0,1} = u_{1,0,0} + \sum_{i=1}^{\infty} v_{0,0}^{(i)} \epsilon^i, \qquad u_{0,0,-1} = u_{-1,0,0} + \sum_{i=1}^{\infty} w_{0,0}^{(i)} \epsilon^i$$

into the initial symmetry. The expansion of this symmetry in a Taylor series gives an infinite number of symmetries.

The application of the Gardner method for the generation of conservation laws for the asymmetric equation

$$\alpha u_{0,0}u_{1,0} - \beta(u_{0,0}u_{0,1} + u_{1,0}u_{1,1}) = 0 \qquad (41)$$

is also considered. In this case we can again mention that all necessary information for the application of the method follows from multidimensional consistency. Equation (41) is consistent with the symmetric equation

$$\alpha(u_{0,0}u_{1,0} + u_{0,1}u_{1,1}) - \beta(u_{0,0}u_{0,1} + u_{1,0}u_{1,1}) = 0. \qquad (42)$$

We obtain a three-dimensional consistent system by imposing (42) on two opposite sides of a cube and (41) on the remaining four sides. From this consistency follows the Lax pair for (41)

$$L = \begin{bmatrix} 1 & \alpha u_{1,0} \\ \dfrac{\alpha\lambda^2}{u_{0,0}} & -\dfrac{u_{1,0}}{u_{0,0}} \end{bmatrix}, \qquad M = \begin{bmatrix} 1 & \beta u_{0,1} \\ \dfrac{\beta\lambda^2}{u_{0,0}} & 0 \end{bmatrix}.$$

Equation (41) satisfies all the criteria of integrability [22], namely there exist infinite number of conservation laws and symmetries, and a Lax pair. As is seen from this paper all these properties are connected with multidimensional consistency. So, multidimensional consistency is a universal criterion of integrability. It is not necessary for an equation to be consistent with itself, as we showed for (41); it can be consistent with other equations.

At this point we do not consider the question of independence of conservation laws and symmetries obtained by the Gardner method. This question requires a deeper understanding of the properties of ABS equations. We are researching this question now and we will publish the results in subsequent articles.

Here are some interesting topics for future research.

- The conditions for the application of the Gardner method to the consistent equation (equation and its BT).
- The classification of quad-graph equations according to these conditions.

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